Problem Sheet 12

Problem 1

Let $(a_n)_n \in \mathbb{Q}_p$ be a series of p-adic numbers $\neq -1$. Show that the infinite product

$$\prod_{n=1}^{\infty}(1+a_n) := \lim_{n\to\infty}\prod_{i=1}^{n}(1+a_i)$$

converges with limit $\neq 0$ if and only if $a_n \to 0$ for $n \to \infty$.

Problem 2

Let K be a field complete with respect to a non-archimedean absolute value and $\mathcal{O} \subseteq K$ its valuation ring. Let $f(X) \in \mathcal{O}[X]$ and $\alpha_0 \in \mathcal{O}$ with $|f(\alpha_0)| < |f'(\alpha_0)|^2$. Using the Newton iteration $\alpha_n = \alpha_{n-1} - \frac{f(\alpha_{n-1})}{f'(\alpha_{n-1})}$ prove that there exists a unique $\alpha \in \mathcal{O}$ with $f(\alpha) = 0$ and

$$|\alpha - \alpha_0| \leq \frac{|f(\alpha_0)|}{|f'(\alpha_0)|} < |f'(\alpha_0)| \leq 1.$$

Show as application that

$$\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2.$$

Problem 3

Let K be a finite extension of $\mathbb{C}((t))$ of degree n; denote by $\mathcal{O}_K$ its valuation ring with respect to the extension of the t-adic valuation.

(a) Show that every $u \in \mathcal{O}_K^\times$ has a d-th root in K, $d \geq 1$.

(b) Prove that $K \cong \mathbb{C}((t^{1/n}))$ as extension of $\mathbb{C}((t))$.

(c) Conclude that the field of Puiseux series $\cup_n \mathbb{C}((t^{1/n}))$ is algebraically closed.

Problem 4

Let $K = \mathbb{F}_p((t))$ and $\mathcal{O}_K = \mathbb{F}_p[[t]]$.

(a) Show that $x^p - x - t$ has p different zeroes in K.

(b) Prove that $f(x) = x^p - x - t^{-1}$ is irreducible over K.